Mathematics and Music

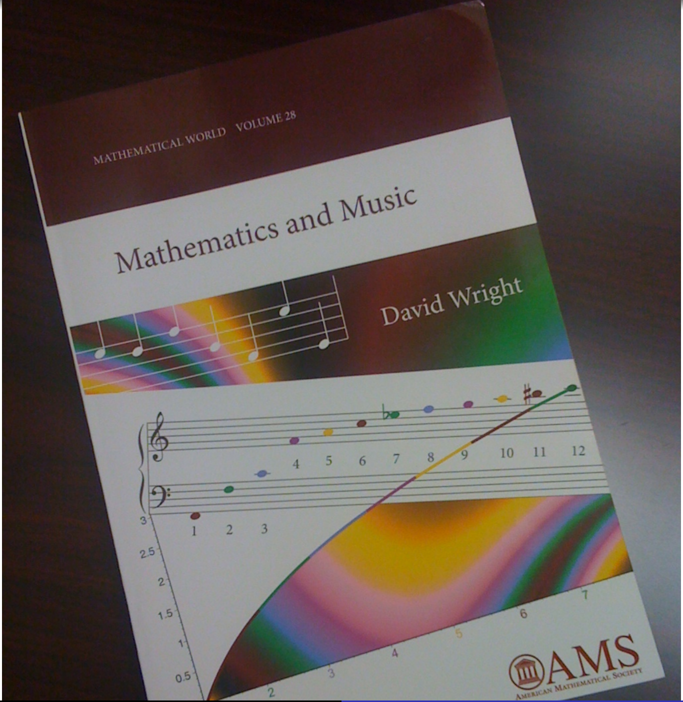
David Wright

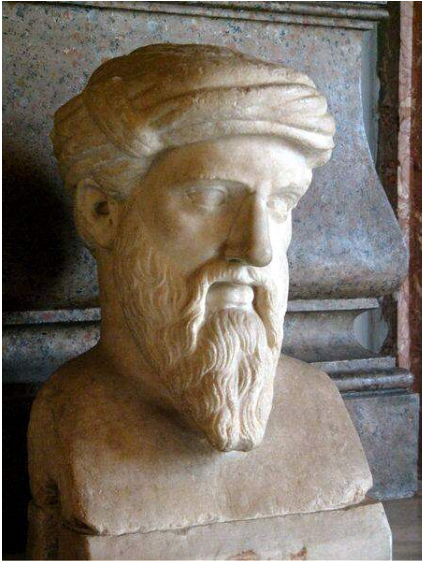
Professor
Department of Mathematics
Washington University
Saint Louis, MO, USA

and

Associate Director
Ambassadors of Harmony
Saint Charles, MO, USA

Field of Dreams Conference
Washington University
Saint Louis, Missouri, November 4-5, 2016


MATHEMATICAL WORLD VOLUME 28
Mathematics and Music
David Wright
AMS
AMERICAN MATHEMATICAL SOCIETY



Pythagoras (c.570 - c.495 BC) was interested in the relationships between harmonious tones.

J. S. Bach (1685-1750) was interested in the mathematical problem of tuning keyboards.

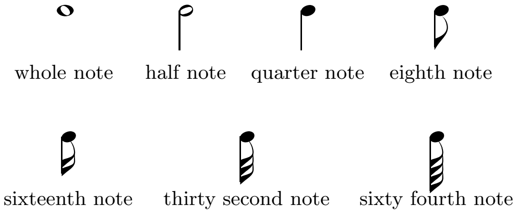
BEGINNING MATH CONCEPTS

- sets, equivalence relations
- functions, graphs
- integers, rational numbers, real numbers
- modular arithmetic
- trigonometry

BEGINNING MUSIC CONCEPTS

- tempo, rhythm
- scales, key signatures
- melody, form
- pitch, intervals, tuning
- tone, timbre

Music's temporal notation is based on powers of two. We divide time intervals in half.



These are names for equivalence classes of notes.

Dots

Music notation's method of extending the duration of a note is by adding dots. If the note 𝅝 has duration d, then

𝅝·	has duration	$\frac{3}{2}d = (1 + \frac{1}{2})d$
𝅝··	has duration	$\frac{7}{4}d = (1 + \frac{1}{2} + \frac{1}{4})d$
𝅝···	has duration	$\frac{15}{8}d = (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8})d$

This hearkens to the geometric series in mathematics:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$$

Meter

Meter is given by a pair $\frac{m}{n}$ where m represents the number of beats in a measure of time, $n = 2^r$ dictates the temporal note which receives one beat.

When counting time the human brain is most comfortable with small primes, reflected in the fact that most time signatures involve 2 and 3.

Audio Example 1   Audio Example 2

Counting in fives and sevens is less common.

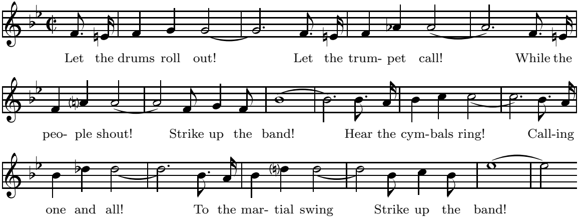
Audio Example 3   Audio Example 3

Melodic transformation

Many songs feature melodic transpositions, analogous to geometric transformations in mathematics. Here are two types:

(1) diatonic

(2) chromatic



Audio Example 3 Audio Example 3

Patterns of m on n in music

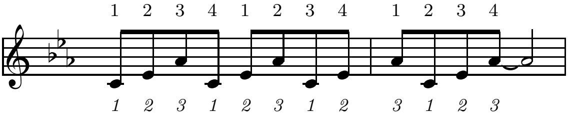
The modular integers, mod n, are the elements of the set

$$\mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\}$$

Modular arithmetic relates to music in several ways. Here is one.

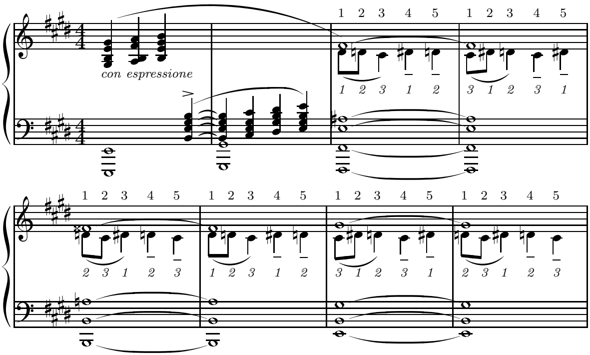
Composers sometimes create ingenious musical passages by imposing a pattern of m notes or beats against a pattern of n such, where $\gcd(m, n) = 1$. This technique exploits (perhaps unknowingly by the composer) the fact that $[m]$ is a generator in $\mathbb{Z}_n$ (and vice versa).

One way this can occur is by cycling m pitches through a repeated rhythmic pattern of n notes. This is exemplified in the main melodic line of the big band song *In the Mood.* Here $m = 3$ and $n = 4$. The song's "hook" lies in the repetition of the rhythmic figure comprising four eighth notes in swing time.



Audio Example

This poignant passage from George Gershwin's *Rhapsody in Blue*, exibits the same phenomenon with $m = 3$, $n = 5$, starting in the third measure.

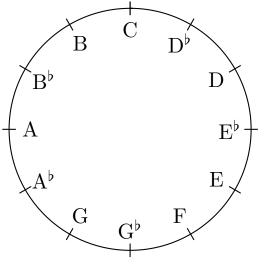


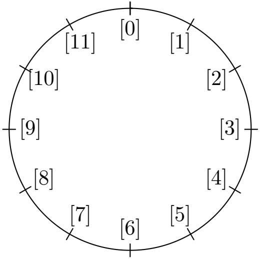
Audio Example 1

Another type of m on n pattern occurs when a melodic figure of duration m beats is repeated in a meter which has the listener counting in groups of n beats. An example of this occurs in the vamp section of the 1971 blues-pop song *Ain't No Sunshine.* Here $m = 3$, $n = 16$.

Audio Example

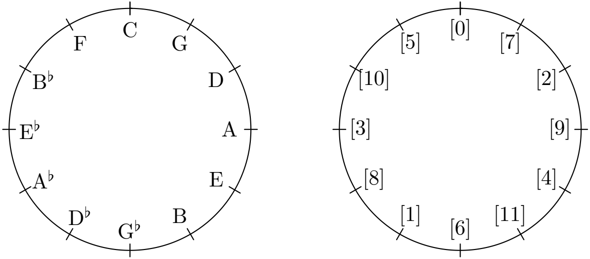
Modular arithmetic makes another entry into music with the chromatic scale. Keyboard intervals, measured in semitones, can be viewed as integers modulo 12, the set of which is denoted $\mathbb{Z}_{12}$.





Circle of fifths

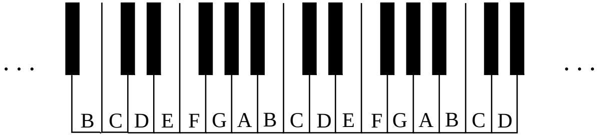
The group generators of $\mathbb{Z}_{12}$ are $[1], [5], [7], [11]$, corresponding to the semitone, fourth, fifth, and major seventh. The middle two give the circle of fifths (and fourths).



Pitch

Pitch is measured in hertz (Hz) (cycles per second). The range of human audibility is roughly 20-20,000 Hz.

Keyboard notes can be parameterized by the integers $\mathbb{Z}$.



However, the set of pitches is a continuum parameterized by the positive real numbers $\mathbb{R}^+$. Many forms of music exploit this continuum.

Audio Example 1 Audio Example 2

The "blue note"

MuddyWaters

Musical intervals

Musical intervals can be measured by:

- semitones s (additive measure)
- ratio r (multiplicative measure)

The translator between the two are the functions:

$$r = 2^{s/12} = \left(\sqrt[12]{2}\right)^s$$

$$s = 12\log_2(r) = \log_{\sqrt[12]{2}}(r)$$

which are inverse to each other.

Microtonal musical intervals are measured in *cents.* One cent is 1/100 of a semitone, so $1200\log_2(r)$ converts ratio to cents.

Example 1: one cent Example 2: 10 cents Example 3: 20 cents

Irrationality of keyboard intervals

Note that the ratio measurement of one semitone is $2^{1/12} = \sqrt[12]{2}$, an irrational number.

In fact: The only rational keyboard intervals are the multi-octaves.

Theorem

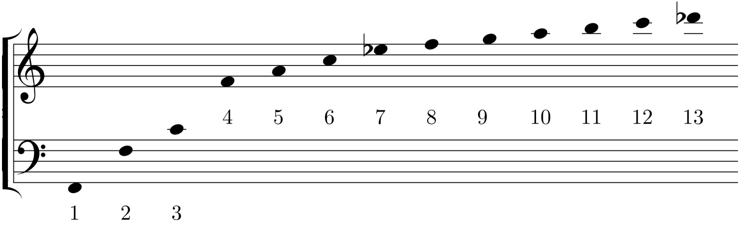
Let I be the interval between two keyboard notes. If I is not an iteration of octaves (i.e. a power of 2 as a ratio), then I is an irrational interval.

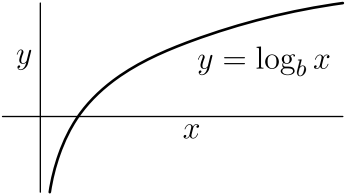
Remark

This theorem would hold for a keyboard that divided the octave up into n equal intervals, for any positive integer n.

Positive integers as intervals

The positive integers $\mathbb{Z}^+$, considered as musical intervals measured as ratios, give an ascending sequence which appears roughly logarithmic when place on a musical staff. (Note: Only the powers of 2 (2,4,8,16, ...) are true keyboard intervals.)





Musical tone

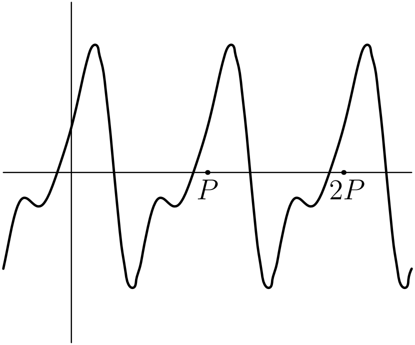
A musical tone at a sustained pitch is a vibration with constant frequency.

English horn playing A 220 Audio Example

Human voice singing “ah" A 220 Audio Example

Periodic functions and musical tone

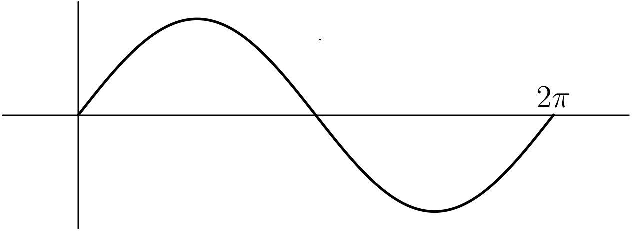
A musical tone is given by an oscillation, or repeating pattern of motion, which is represented by a periodic function.



The number P is called the *period* of the function. If P is measured in seconds, then the *frequency*, or *pitch*, of the tone, given by $F = 1/P$, is measured in cycles per second, or *hertz* (Hz).

The sine wave

The most basic periodic function is the $f(t) = \sin t$, which represents the simple up-and-down motion of a weight hanging on a spring. Its graph is the familiar sine wave:



The sound it generates is a dull hum: Audio Example

Fourier series

Theorem

Suppose $f(t)$ is periodic of period 2π which is bounded and has a bounded continuous derivative at all but finitely many points in $[0, 2\pi)$. Then there is a real number C and sequences of real numbers $A_1, A_2, A_3 \dots$ and $B_1, B_2, B_3 \dots$ such that, for all t at which $f(t)$ is continuous we have $f(t)$ given by the sum

$$f(t) = C + \sum_{k=1}^{\infty} \left[A_k \sin(kt) + B_k \cos(kt) \right] .$$

Fourier coefficients

The coefficients appearing in

$$f(t) = C + \sum_{k=1}^{\infty} \left[A_k \sin(kt) + B_k \cos(kt) \right] .$$

are given by these formulas:

$$\begin{aligned} C &= \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt \\ A_k &= \frac{1}{\pi} \int_0^{2\pi} \sin(kt) f(t)\, dt \\ B_k &= \frac{1}{\pi} \int_0^{2\pi} \cos(kt) f(t)\, dt \end{aligned} \tag{1}$$

These are called *Fourier coefficients*.

Harmonics

Now consider a periodic function $g(t)$ of arbitrary frequence F. An application of the $\sin(\alpha + \beta)$ formula yields:

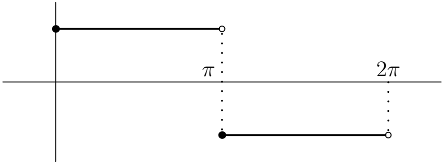
$$g(t) = C + \sum_{k=1}^{\infty} d_k \sin(2\pi F k t + \beta_k)\,.$$

k - the index of the *harmonic* having frequency kF
d_k - the "weight" of the k^{th} harmonic (important!)
β_k - the phase shift of the k^{th} harmonic (not important here)

The relative weights d_k determine the *timbre* of the tone, allowing us to distinguish between different musical instruments and different human vowel sounds.

Audio Example

$$s(t) = \begin{cases} 1, & \text{for } 0 \le t < \pi \\ -1, & \text{for } \pi \le t < 2\pi \end{cases}$$



Using the integral formulas one can calculate:

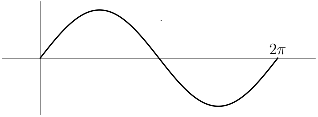
$$C = 0, \qquad B_k = 0 \text{ for all } k, \qquad A_k = \begin{cases} 0, & \text{for } k \text{ even} \\ \frac{4}{k\pi}, & \text{for } k \text{ odd} \end{cases}$$

hence

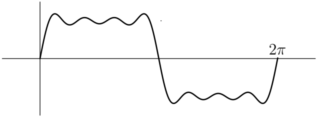
$$s(t) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right)$$

It sounds like: Audio Example

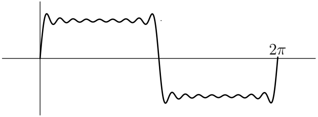
Partials sums for the square wave



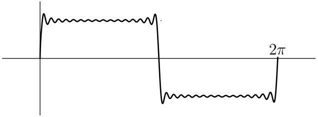
first summand



first 3 summands (1,3,5)



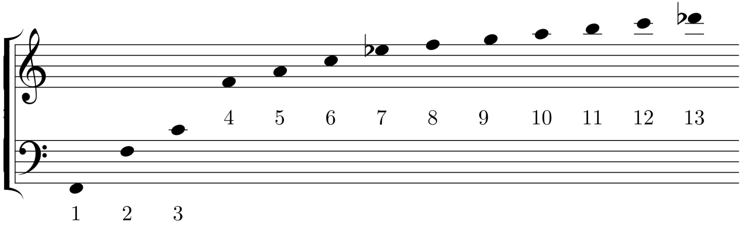
first 8 summands (1,3,. . .,15)



first 15 summands (1,3,. . .,29)

Here are the sounds as we add one harmonic at a time:
1   1,3   1,3,5   1,3,5,7   1,3,5,7,9   1,3,5,7,9,11   1,3,5,7,9,11,13

Remember that the positive integers $\mathbb{Z}^+$ represent musical intervals measured as ratios from a fixed pitch (here F_2).



This sequence of tones are the harmonics (overtones) of the lowest note.

Except for powers of 2, these keyboard notes are inexact approximations. Note, for example, that the keyboard approximates 6/5 and 7/6 by the same interval – the keyboard minor third.

Rational intervals

The interval from the k^{th} harmonic to the ℓ^{th} harmonic is $\ell : k$, or ℓ/k. This accounts for all rational intervals.

The most basic interval is the octave, whose ratio is $2 : 1$, reflecting the fact that the brain instinctively comprehends 2. Sometimes we have trouble distinguishing notes an octave apart.

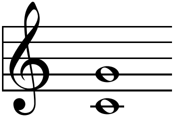
Audio Example

Music frequently uses *octave equivalence,* which declares keyboard notes to be equivalent if the interval between them is n octaves, for $n \in \mathbb{Z}$. The note names C, E$^\flat$, G$^\sharp$, etc., are actually equivalence classes.

The rational numbers which are the ratios between the lower pitches in the harmonic series give us the "true", or *just*, intervals of music. For example, the ratio 3 : 2 is the just fifth, which is accurately, but not precisely, rendered on the keyboard, according to the computation:

$$1200 \log_2(3/2) \approx 701.955 \text{ cents}$$

The keyboards fifth is seven semitones, or 700 cents.



approximation of $\frac{3}{2}$ ($\approx$ 2 cents flat)

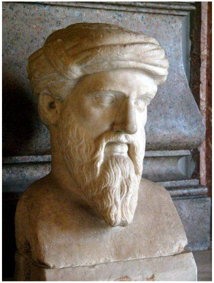
Audio Example 1 Audio Example 2

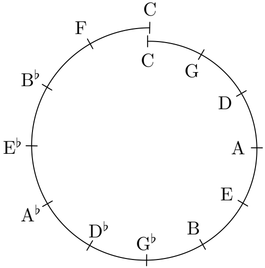
Comma of Pythagoras

The just fifth was the source of frustration for Pythagoras, who wanted to tune the scale around fifths. The overshoot of 12 just fifths over 7 octaves is:

$$\frac{(3/2)^{12}}{2^7} = \frac{3^{12}}{2^{19}} = \frac{531441}{524228} \approx 1.01364326$$

which is $1200 \log_2((3^{12}/2^{19}) \approx 23.46$ in cents. This is called the *comma of Pythagoras*. (The tempered scale shrinks the fifths.)





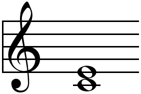
7-octave clock with just fifths

Major third

The ratio 5 : 4 gives the just major third, which is measured in cents by

$$1200 \log_2(5/4) \approx 386.314 \text{ cents}$$

whereas the keyboard's major third is four semitones, or 400 cents.



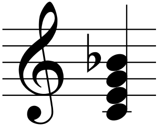
approx. of $\frac{5}{4}$ ($\approx$ 14 cents sharp)

The keyboard's third is audibly sharp.

Audio Example 1   Audio Example 2

Seventh

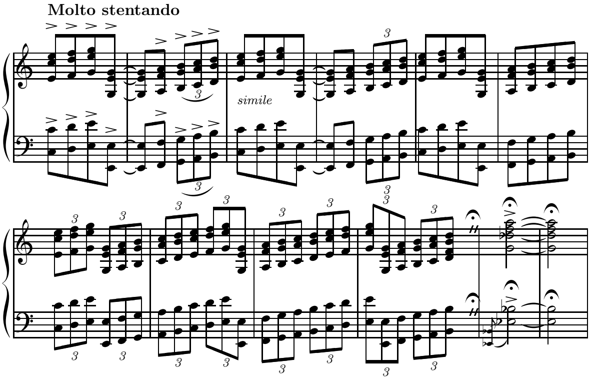
The just rendition of the (dominant) seventh chord has ratio $4:5:6:7$.



The just, or *septimal*, seventh is audibly different from the keyboard seventh.

Audio Example 1 Audio Example 2

The striking six-note final chord can be tuned justly as $2:3:5:7:9:11$, whereupon all these notes occur as harmonics of $E\flat_2$. Though the 7 and 11 are poorly approximated by equal temperament, the chord's primal appeal likely comes from its similarity to the just rendition. Here is a striking example:

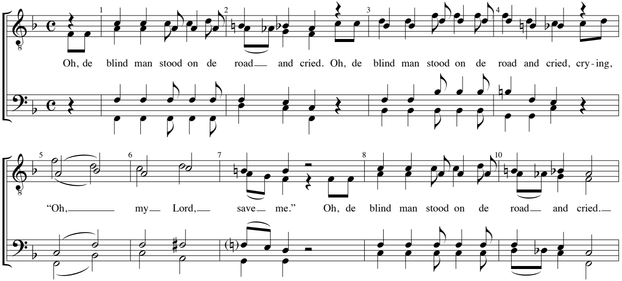


Audio Example

Real-time tuning

Singers and instumentalists whose instruments can bend pitch (e.g., unfreted stringed instruments) are free to use portamento and to tune by ear in real time. They generally gravitate toward just intonation. Here is a striking example:

De Blind Man Stood On De Road And Cried

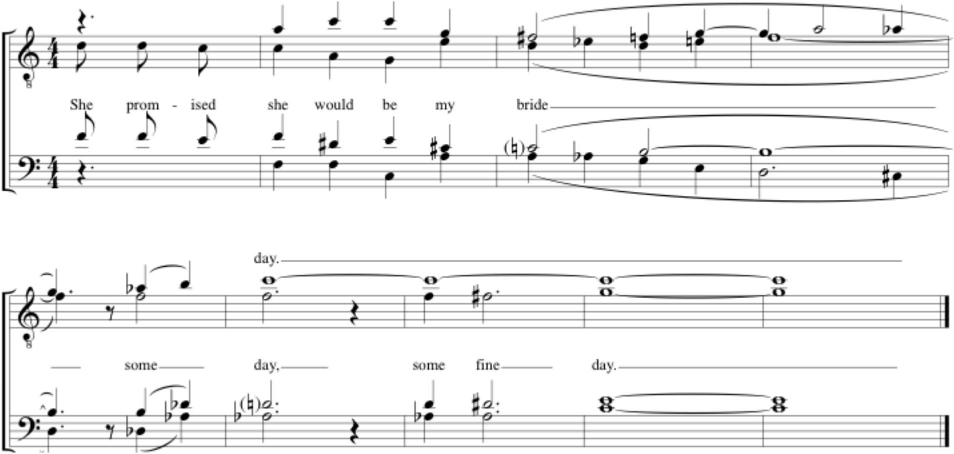


Audio Example

Dover String Quartet

Quartet in B flat major, Op. 76, No. 4 “Sunrise” (Haydn)

Barbershop harmony



Barbershop quartet: The Gas House Gang

Gas House Gang

Bright Was the Night

THANK YOU

and thanks to the

NATIONAL ALLIANCE for DOCTORAL STUDIES

in the MATHEMATICAL SCIENCES

for this invitation.