Probability of Consensus in Spatial Opinion Models with Confidence Threshold

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OUTLINE

Basic Voter Model

Graph Theory – tools

General Opinion Model with Confidence Threshold

Modified Opinion Dynamics with Confidence Threshold

Imitation & Attraction Models
Basic Voter Model

Markov chain $\eta_t : \mathbb{Z} \to \{0, 1\}$

All opinions are equally likely
Each individual mimics a randomly chosen neighbor at rate one
Basic Voter Model

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The (geodesic) distance from $a$ to $b$, $d(a, b) = 4$
Graph Theory – tools

\[ G = (V, E) \]

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eccentricity \( \epsilon \) of a vertex \( v \)
Graph Theory – tools

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The (geodesic) distance from \( a \) to \( b \), \( d(a, b) = 4 \)
eccentricity \( \epsilon \) of a vertex \( v \)
radius \( r = 2 \), diameter \( d = 4 \)
Graph Theory – example

radius $r = 1$, diameter $d = 2$
Graph Theory – example

radius $r = 1$, diameter $d = 2$
General Opinion Model with Confidence Threshold $\tau$

$G = \mathbb{Z} = (\mathcal{V}, \mathcal{E})$ is a spatial graph

Markov chain $\xi_t : \mathbb{Z} \rightarrow \mathcal{V}$,
where $\mathcal{V}$ is the vertex set of the opinion graph $G = (\mathcal{V}, \mathcal{E})$. 
General Opinion Model with Confidence Threshold $\tau$

$\mathcal{G} = \mathbb{Z} = (\mathcal{V}, \mathcal{E})$ is a spatial graph

Markov chain $\xi_t : \mathbb{Z} \rightarrow \mathcal{V}$,
where $\mathcal{V}$ is the vertex set of the opinion graph $G = (\mathcal{V}, E)$.

Individuals interact if and only if their opinion distance $d(a, b) \leq \tau$
General Opinion Model – example

Let $\tau = 2$

Interaction of G and B in $G$

![Diagram](image-url)
General Opinion Model – example

Let $\tau = 2$

Interaction of $G$ and $B$ in $G$

Below is $G$, the opinion graph

$$3 = d(\xi(G), \xi(B)) > \tau \implies \text{no interaction}$$
General Opinion Model – example

\( \tau = 2 \)

Interaction of E and A
General Opinion Model – example

\[ \tau = 2 \]

Interaction of E and A

\[ 2 = d(\xi(E), \xi(A)) = \tau \implies \text{interaction} \]
Imitation Model

Each individual imitates a randomly chosen neighbor at rate one. Individuals interact if and only if their opinion distance is at most $\tau$ ($= 2$).
Imitation Model

Markov chain $\xi_t : \mathcal{V} \rightarrow \mathcal{V}$

Each individual imitates a randomly chosen neighbor at rate one

Individuals interact if and only if their opinion distance is at most $\tau (= 2)$
Imitation Model

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Imitation Model

We define the process

$$X_t = \sum_{x \in \mathcal{V}} 1\{\epsilon(\xi_t(x)) \leq \tau\} = |\{x \in \mathcal{V} : \epsilon(\xi_t(x)) \leq \tau\}|,$$

that keeps track of the number of individuals whose opinion has eccentricity $\epsilon \leq \tau$. 
Lemma. Time to fixation $T$ is an almost surely finite stopping time

Lemma. $(X_t)$ martingale

Optional Stopping Theorem to $(X_t)$

$P(\xi_T \equiv \text{consensus}) > 0$
Imitation Model

\[ \tau \geq d \]

\[ P(\xi_T \equiv \text{consensus}) = 1 \]
Imitation Model

\[ \tau \geq d \]

\[ P(\xi_T \equiv \text{consensus}) = 1 \]

\[ \tau \in [r, d) \]

\[ P(\xi_T \equiv \text{consensus}) \geq \frac{|\{ a \in V : \epsilon(a) \leq \tau \}|}{|V|} > 0 \]
Attraction Model

Each individual moves one opinion distance closer to a randomly chosen neighbor at rate one. Individuals interact if and only if their opinion distance is at most $\tau = 2$. 
Attraction Model

Markov chain $\zeta_t : \mathcal{V} \rightarrow \mathcal{V}$

Each individual moves one opinion distance closer to a randomly chosen neighbor at rate one.

Individuals interact if and only if their opinion distance is at most $\tau (= 2)$.
Attraction Model

Markov chain $\zeta_t : \mathcal{V} \rightarrow \mathcal{V}$

Each individual moves one opinion distance closer to a randomly chosen neighbor at rate one

Individuals interact if and only if their opinion distance is at most $\tau (= 2)$
Attraction Model

The opinion graph of our model is acyclic since our result follows

Lemma (eccentricity inequalities)

\[ \epsilon_{i'} + \epsilon_{j'} \leq \epsilon_i + \epsilon_j \]
Attraction Model

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A non-example of a cyclic opinion graph
The opinion graph of our model is acyclic since our result follows

**Lemma (eccentricity inequalities)**

\[ \epsilon_{i'} + \epsilon_{j'} \leq \epsilon_i + \epsilon_j \]

A non-example of a cyclic opinion graph

\[ \epsilon_i = 102 = \epsilon_j; \quad \epsilon_{i'} = 103 = \epsilon_{j'} \]

This implies that \[ \epsilon_{i'} + \epsilon_{j'} \not\leq \epsilon_i + \epsilon_j \]
Attraction Model

We define the process

\[(Z_t) = \sum_{x \in V} (\epsilon(\zeta_t(x)) - r) = \sum_{a \in V} (\epsilon(a) - r) |\{x \in V : \zeta_t(x) = a\}|,\]

that keeps track of the eccentricity of the individuals’ opinions.
Attraction Model – blueprint

Eccentricity inequality satisfied

Lemma. \((Z_t)\) supermartingale

Lemma. Time to fixation \(T\) is an almost surely finite stopping time

Optional Stopping Theorem to \((Z_t)\)

\[ P(\zeta_T \equiv \text{consensus}) > 0 \]
Attraction Model

$\rightarrow \quad \tau \geq d$

$P(\zeta_T \equiv \text{consensus}) = 1$
Attraction Model

\[ \tau \geq d \]

\[ P(\zeta_T \equiv \text{consensus}) = 1 \]

\[ \tau \in [r, d) \]

\[ P(\zeta_T \equiv \text{consensus}) \geq 1 - \frac{1}{|V|} \sum_{a \in V} \left( \frac{\epsilon(a) - r}{\tau + 1 - r} \right) \]
Attraction Model – $G$: full $n$-ary tree
Attraction Model – $G$: full $n$-ary tree

$\tau \in [r, 2r)$:

$$P(\zeta_T \equiv \text{consensus}) \geq 1 - \left(\frac{1}{\tau + 1 - r}\right) \left(\frac{n(rn^{r+1} - (r + 1)n^{r} + 1)}{(1 - n)(1 - n^{r+1})}\right)$$
Attraction Model – $G$: star-like graph
Attraction Model – $G$: star-like graph

$\tau \in [r, 2r)$:

$$P(\zeta_T \equiv \text{consensus}) \geq 1 - \left( \frac{1}{\tau + 1 - r} \right) \left( \frac{r(r + 1)n}{2(1 + rn)} \right)$$
Thank you!