What is a cone?

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Roadmap for today

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3. Hyperplane Description
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Intuitive idea of a Cone

"Set of vectors closed under positive combinations"

Example

For $V = \{(1, \frac{1}{2}), (1, 2), (2, 1), (\frac{1}{2}, \frac{3}{4})\}$
Intuitive idea of a Cone

"Set of vectors closed under positive combinations"

Example

For \( V = \{ (1, \frac{1}{2}), (1, 2), (2, 1), (\frac{1}{2}, \frac{3}{4}) \} \), the cone of \( V \) is

\[
C(V) = \left\{ a_1 \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} + a_2 (1, 2) + a_3 (2, 1) + a_4 \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \end{pmatrix} \mid a_i \in \mathbb{R}_{\geq 0} \right\}
\]
Vertex/Ray Description

"The space generated by a finite set of vertices/rays"

- Let $V = \{v_1, v_2, \ldots, v_i, r_{i+1}, \ldots, r_m\}$ be a set of vertices and rays in $\mathbb{R}^n$.
- The cone generated by $V$ is

$$C(V) = \{\lambda_1 v_1 + \cdots + \lambda_m r_m \mid \lambda_i \in \mathbb{R}_{\geq 0}\}.$$
Vertex/Ray Description

“The space generated by a finite set of vertices/rays"

Let $V = \{v_1, v_2, \ldots, v_i, r_{i+1}, \ldots, r_m\}$ be a set of vertices and rays in $\mathbb{R}^n$.

The cone generated by $V$ is

$$C(V) = \{\lambda_1 v_1 + \cdots + \lambda_m r_m \mid \lambda_i \in \mathbb{R}_{\geq 0}^n\}.$$
Hyperplane Description

“The intersection of halfspaces”

\[ H_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid \frac{1}{2}x_1 - x_2 \leq 0 \right\} \]

Definition

- A hyperplane \( H \) is the set \( \{x \in \mathbb{R}^n \mid a(x) = 0\} \), for linear map \( a \) over \( \mathbb{R}^n \).
- A closed halfspace \( H \) is choosing a “side” of \( H \):

\[ \{x \in \mathbb{R}^n \mid a(x) \geq 0\} \].
Hyperplane Description

"The intersection of halfspaces"

\[ \mathcal{H}_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid \frac{1}{2}x_1 - x_2 \leq 0 \right\} \]

\[ \mathcal{H}_2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 - x_2 \geq 0 \right\} \]

Definition

- A hyperplane \( H \) is the set \( \{x \in \mathbb{R}^n \mid a(x) = 0\} \), for linear map \( a \) over \( \mathbb{R}^n \).
- A closed halfspace \( \mathcal{H} \) is choosing a "side" of \( H \):
  \[ \{x \in \mathbb{R}^n \mid a(x) \geq 0\} \]
Hyperplane Description

“The intersection of halfspaces”

Definition

A convex cone $C$ is a collection of closed halfspaces $A$, such that $C = \{x \in \mathbb{R}^n | Ax \leq 0\}$. 
Theorem (Weyl–Minkowski Theorem)

A convex polyhedral cone has both a vertex/ray and hyperplane description, which are equivalent.
Where Cones Commonly Show Up

- Solvability of a general system of linear equations (Farka’s lemma)
- Integer point enumeration, Ehrhart Theory
- Discrete optimization, linear programming, feasibility problems
- Computational Complexity

*Where else might they show up?*
Using Cones to Understand Graphs

Definition

- A graph $G = (V, E)$ is a set of vertices and edges.
- A cycle of $G$ is a set of edges forming a path that returns to itself only once.

Example
Using Cones to Understand Graphs

We can describe all the cycles of a graph using vectors!

- Let $c \in \{0, 1\}^n$ be the indicator vector of a cycle of graph $G$, where $c_i = 1$ if $e_i \in E$ and 0 if not.

Example

Cycle in $G = (1, 0, 1, 0, 1, 1)$
Using Cones to Understand Graphs

We can describe all the cycles of a graph using vectors!

- Let $c \in \{0, 1\}^n$ be the indicator vector of a cycle of graph $G$, where $c_i = 1$ if $e_i \in E$ and 0 if not.

Example

\[
C = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

- Every column is a cycle and rows are indexed by edges.
Using Cones to Understand Graphs

Using the set of cycles of $G$, we can generate the cone $C_G$ over all cycles of $G$:

$$C_G = \{ \lambda_1 c_1 + \cdots + \lambda_n c_n \mid \lambda \in \mathbb{R}^n \}$$

**Example**

$$C_G = \left\{ \begin{array}{c}
\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \cdots + \lambda_7 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\
\lambda_i \in \mathbb{R}^7
\end{array} \right\}$$
Using Cones to Understand Graphs

Why use cones? For a new perspective!

- **CDC conjecture:** For any graph $G$, there exists a set of cycles covering the edges of $G$ so that every edge is in exactly 2 cycles.
Using Cones to Understand Graphs

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![Graph example](image)
Using Cones to Understand Graphs

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Using Cones to Understand Graphs

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  \[
  \{ (2, 2, \ldots, 2) \}
  \]

  \[\begin{array}{c}
  V_1 \\
  V_2 \\
  V_3 \\
  V_4 \\
  \end{array}
  \]

  \[\begin{array}{c}
  E_1 \\
  E_2 \\
  E_3 \\
  \end{array}
  \]

  \[\begin{array}{c}
  C_1 \\
  C_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  D_1 \\
  D_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  F_1 \\
  F_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  G_1 \\
  G_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  H_1 \\
  H_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  I_1 \\
  I_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  J_1 \\
  J_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  K_1 \\
  K_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  L_1 \\
  L_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  M_1 \\
  M_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  N_1 \\
  N_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  O_1 \\
  O_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  P_1 \\
  P_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  Q_1 \\
  Q_2 \\
  \end{array}
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  \[\begin{array}{c}
  R_1 \\
  R_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  S_1 \\
  S_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  T_1 \\
  T_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  U_1 \\
  U_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  V_1 \\
  V_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  W_1 \\
  W_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  X_1 \\
  X_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  Y_1 \\
  Y_2 \\
  \end{array}
  \]

  \[\begin{array}{c}
  Z_1 \\
  Z_2 \\
  \end{array}
  \]

Via Cones: *The integral cone of cycles of $G$ always contains $(2, 2, \ldots, 2)$.*
Using Cones to Understand Graphs

Why use cones? For a new perspective!

- **CDC conjecture**: For any graph $G$, there exists a set of cycles covering the edges of $G$ so that every edge is in exactly 2 cycles.

Via Cones: *The integral cone of cycles of $G$ always contains $(2, 2, \ldots, 2)$.*

- **In general**: Given vector $u = (u_1, u_2, \ldots, u_n)$, is there a set of cycles so that edge $i$ is covered $u_i$ many times?
Using Cones to Understand Graphs

Why use cones? For a new perspective!

- **CDC conjecture:** For any graph $G$, there exists a set of cycles covering the edges of $G$ so that every edge is in exactly 2 cycles.

  ![Graph Illustration]

  Via Cones: *The integral cone of cycles of $G$ always contains $(2, 2, \ldots, 2)$.***

- **In general:** Given vector $u = (u_1, u_2, \ldots, u_n)$, is there a set of cycles so that edge $i$ is covered $u_i$ many times?

  *Does the integral cone of cycles of $G$ contain $u$?*
References


About me

Instinct

Play time!

Community Support

Allies and Cheerleaders

The Future
Thank you!

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Play time!