What is a convex body?

Galyna V. Livshyts

Mathematical Sciences Research Institute,
Berkeley, CA

Field of dreams, Saint Louis, MO
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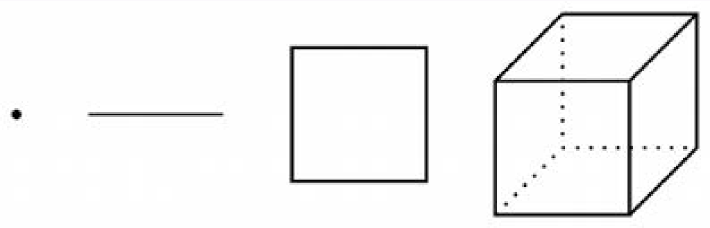
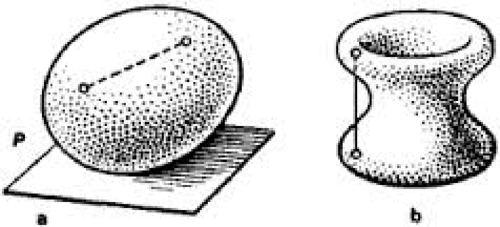


Convex set

A set K in $\mathbb{R}^n$ is called convex if for any pair of points $x, y \in K$, the interval $[x, y]$ is fully contained in K.



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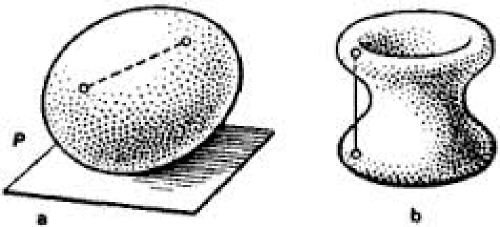

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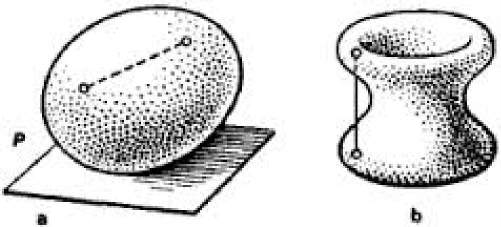

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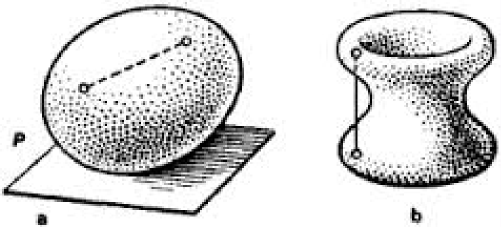

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- A body K is called symmetric if $x \in K \implies -x \in K$.

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- Convex hull of N points $x_1, ..., x_N$

$$conv(x_1, ..., x_N) = \{y \in \mathbb{R}^n : y = \lambda_1 x_1 + ... + \lambda_N x_N, \lambda_i \geq 0, \sum \lambda_i = 1\}.$$

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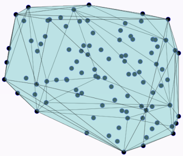

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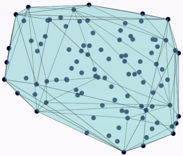

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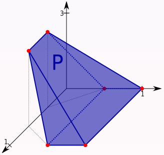

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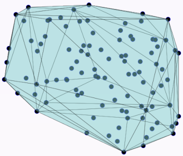

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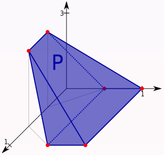

- Moreover, every convex body is an intersection of (very many) half-spaces, as well as the convex hull of its boundary points.

CUBE

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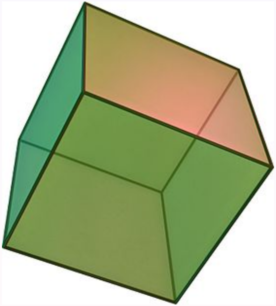

Examples of convex bodies

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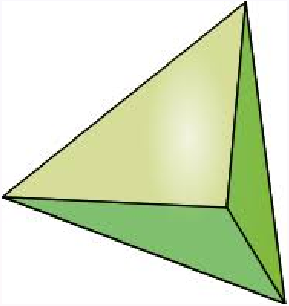

DIAMOND

The diamond, or cross-polytope is the set defined as

$$B_1^n = \{x \in \mathbb{R}^n : |x_1| + \ldots + |x_n| \le 1\}.$$

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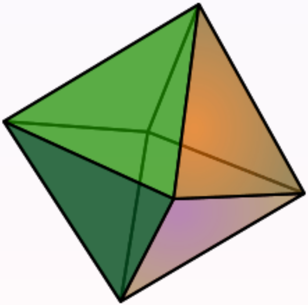

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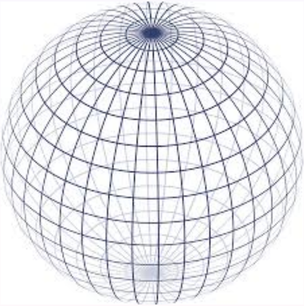

L_p-BALL

More generally, for $p \geq 1$, L_p-ball in $\mathbb{R}^n$ is the set

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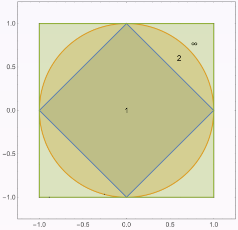

- $p = 2$ – usual "euclidean" ball;
- $p = 1$ – cross-polytope;
- $p = \infty$ – cube!

Ball, cube and diamond are very far!

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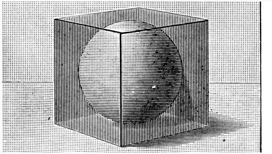

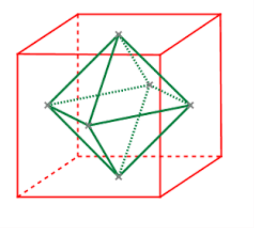

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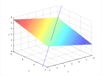
- $$Vol_n(B_2^n) = \left(\frac{\sqrt{2\pi e}}{\sqrt{n}} \right)^n;$$
- The unit ball gets smaller and smaller!
- The radius of the ball of unit volume is of order $\sqrt{n}$!

Hyperplanes

For a unit vector ξ, define the hyperplane $H = \xi^{\perp}$, orthogonal to ξ, as

$$\xi^{\perp} = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}.$$



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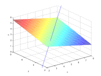
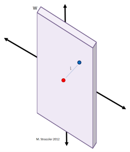


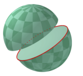
Slabs

For a unit vector ξ, define the slab S_ξ, orthogonal to ξ, of width ρ, as

$$S_\xi = \{x \in \mathbb{R}^n : |\langle x, \xi \rangle| \leq \rho\}.$$



- Note that $\frac{Vol_{n-1}(B_2^{n-1} \cap \xi^{\perp})}{Vol_n(B_2^n \cap \xi^{\perp})} = c\sqrt{n}$ – large!



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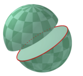

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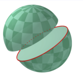

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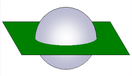

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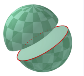

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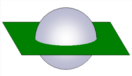
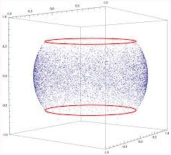


- Therefore, the constant portion of the volume of the ball is contained in a slab of constant width!



More on distribution of mass in the ball

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Denote the boundary of the unit ball by $\mathbb{S}^{n-1}$ – unit sphere. Integrating in polar coordinates, we get

$$Vol_n(B_2^n) = \int_{\mathbb{S}^{n-1}} \int_0^1 t^{n-1} dt d\theta = \frac{|\mathbb{S}^{n-1}|}{n}.$$

Note that

$$\int_0^1 t^{n-1} = const \cdot \int_{1-\frac{1}{n}}^1 t^n dt.$$

We know that most of the mass of the ball comes from thin slabs. Does it mean that the volume is concentrated in the center?..
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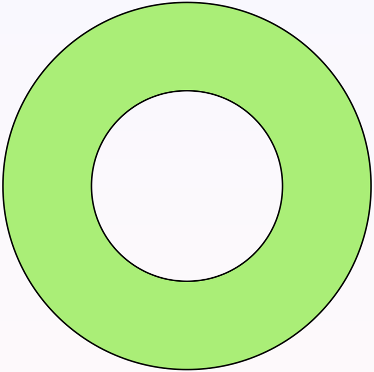
Therefore, a constant portion of the volume of the unit ball is in the thin spherical shell near the boundary!

More on distribution of mass in the ball



Smallest section of the unit cube

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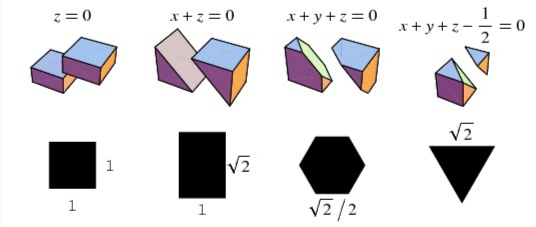

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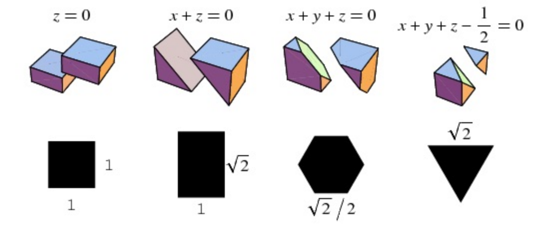
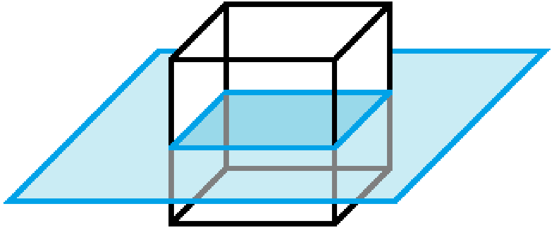


Turns out that the smallest (in area) section of the unit cube is the one parallel to coordinate subspaces:



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Theorem (Keith Ball, 1984)

For every dimension n and for every unit vector $u \in \mathbb{R}^n$,

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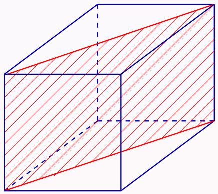
This estimate is sharp!



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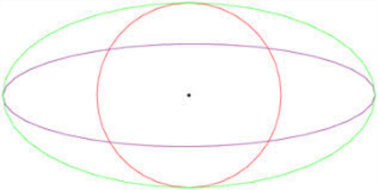
This question is wide open! The best known estimate is due to Klartag, and it says that the largest section of a convex body of unit volume cannot be smaller than $cn^{-\frac{1}{4}}$.



Central Limit Theorem

Let $X_1, \ldots, X_n$ be independent identically distributed random variables. Then $\frac{X_1+\ldots+X_n}{\sqrt{n}}$ has almost normal (gaussian) distribution.

Central Limit Theorem and the cube

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Example: uniform random variables on $[0, 1]$.

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$$\frac{X_1 + ... + X_n}{\sqrt{n}} = \langle X, u \rangle,$$

where $u = (\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}})$ and X is a random vector uniformly distributed in the unit cube.

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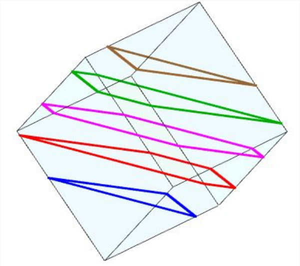
where $u = (\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}})$ and X is a random vector uniformly distributed in the unit cube.
- The density of $\langle X, u \rangle$ takes values which are hyperplane sections of the cube, orthogonal to u!

CLT and the cube

The Central Limit Theorem tells us that the sections of the cube have almost Gaussian distribution!



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- Thus, all the convex bodies are a bit like cubes!
- Also, all the convex bodies are a bit like balls: for every convex body in $\mathbb{R}^n$ there is a section (of small dimension) which looks very much like a ball! This is the content of Milman-Dvoretszki theorem.
- But we said that balls and cubes are very far from each other... Oops!

Thanks for your attention!